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Riemann–Liouville integrals of fractional order and extended KP hierarchy

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Abstract
An attempt to formulate the extensions of the KP hierarchy by introducing fractional-order pseudo-differential operators is given. In the case of the extension with the half-order pseudo-differential operators, a system analogous to the supersymmetric extensions of the KP hierarchy is obtained. Unlike the supersymmetric extensions, no Grassmannian variable appears in the hierarchy considered here. More general hierarchies constructed by the $1/N$th-order pseudo-differential operators, their integrability and the reduction procedure are also investigated. In addition to finding the new extensions of the KP hierarchy, a brief introduction to the Riemann–Liouville integral is provided to yield a candidate for the fractional-order pseudo-differential operators.

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1. Introduction

Integrable hierarchies of nonlinear partial differential equations (PDEs) have been vigorously studied from the perspective of physics as well as mathematics. Among them the Kadomtsev–Petviashvili (KP) hierarchy and its variants appear in many areas of theoretical physics. In particular, the supersymmetric extensions of the KP hierarchy play important roles in non-perturbative superstring theories [1], and connections are suggested between the dispersionless limit of the KP hierarchy and topological field theory [2, 3].

Concerning the construction of the KP hierarchy in the Lax formalism, the non-commutative algebra of pseudo- or micro-differential operators plays a fundamental role [4, 5]. For the standard KP hierarchy, the associated pseudo-differential operator can be regarded as an ordinary integral operator, which enjoys the generalized Leibniz rule. The aim of this paper is to enquire into the practicability of extensions of the KP hierarchy by introducing

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fractional-order pseudo-differential operators. In this respect, we recall that the survey of the fractional-order integration and differentiation is known as fractional calculus.

The fractional calculus, which usually stands for the differentiation and integration of arbitrary order so that the terminology is somewhat misleading, has a long and rich history [6, 7]. The standard definition of the arbitrary-order integration/differentiation is mostly given by the so-called Riemann–Liouville integral these days. Although the fractional calculus has been well studied in mathematics, it is not an ordinary mathematical tool in the theory of integrable systems at present. Apart from integrable systems, there are many applications of fractional calculus in physics; for example, one of the present authors analysed the supersymmetric field theories through half-order differential operators [8], other important work in the subject has been performed on non-differential evolution equations, chaotic dynamical systems, material physics and so on [9].

In the present paper, we consider extensions of the KP hierarchy by introducing the fractional-order integral/differential operators as pseudo-differential operators, which should be interpreted as the power roots of ordinary integration/differentiation; the situation is similar to the supersymmetric extensions of the KP hierarchy [10–13], where the square roots of integral/differential operators are brought in through superspace formulation. In contrast, we extend the KP hierarchy by making use of fractional-order integral/differential operators with respect to purely 'bosonic' variables, for which the relevant non-commutative algebra is the generalized Leibniz rule of fractional order. We will see in the following that the extension of the KP hierarchy by half-order integral/differential operators leads to a hierarchy similar to that of supersymmetric extension, as expected.

This paper is organized as follows. In the next section, we give a very brief review of the Lax formulation of the KP hierarchy and its supersymmetric extension for the purpose of determining notation. In section 3, we make an attempt to generalize the KP hierarchy by fractional-order integral/differential operators and find the formulation works consistently. In section 4, we introduce the Riemann–Liouville integrals as a candidate for the pseudo-differential operators of fractional order, which supply the generalized Leibniz rule being used in section 3. The final section is devoted to concluding remarks.

2. The Lax formulation of the KP hierarchy

In this section we give a sketch of the Lax formulation of the standard KP hierarchy, its k-reduction and supersymmetric extensions, to fix the notation throughout the present paper.

2.1. The standard KP hierarchy

The KP hierarchy within the framework of the Lax formulation is generated by the non-commutative algebra of the pseudo-differential operator ∂^{-j} with respect to an independent variable x, which acts on a function through the generalized Leibniz rule

$$\partial^{-j} \circ f = \sum_{k=0}^{\infty} \binom{-j}{k} f^{(k)} \partial^{-j-k}. \tag{1}$$

Here, we consider the case of integer j, the order of pseudo-derivative or integral, although formula (1) is valid for non-integer j. We define the Lax operator of the (one-component) KP hierarchy by

$$\mathcal{L}_{\mathrm{KP}} = \partial + \sum_{j=1}^{\infty} u_{j+1} \partial^{-j} \tag{2}$$

where u_j are dependent variables of space x and time variables being introduced below. The coefficient of ∂^0 can be set to zero without loss of generality. We assign the degree of the differential operator ∂ as one, standing for $\deg[\partial] = 1$, and assume that all the terms in the Lax operator (2) have equal degree, i.e. $\deg[u_j] = j$. This assignment of the degree is naturally justified by the tau-function formalism [4]. Introducing infinite directions of 'time' $\mathbf{t} = (t_1, t_2, t_3, \ldots)$ with $\deg[t_n] = -n$, we may consider the Lax equations

$$\frac{\partial \mathcal{L}_{\mathrm{KP}}}{\partial t_n} = [\mathcal{B}_n, \mathcal{L}_{\mathrm{KP}}] \qquad (n = 1, 2, 3, \ldots). \tag{3}$$

If we define the nth 'Hamiltonian' $\mathcal{B}_n$ by the non-negative power part in ∂ of the nth product of the Lax operator (2), denoting $\mathcal{B}_n := \left(\mathcal{L}_{\mathrm{KP}}^n\right)_+ = \left(\mathcal{L}_{\mathrm{KP}}^n\right)_{\geqslant 0}$, we will obtain an infinite tower of nonlinear PDEs, the standard KP hierarchy. Note that the lowest time variable t_1 should be identified with the space variable x due to the first Lax equation. The lowest PDE, the KP equation, is obtained by comparing the coefficients of ∂^{-j} on each side of (3) for t_2 and t_3 developments of u_2, u_3 and u_4 and eliminating the u_3 and u_4,

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu'\right]' \tag{4}$$

where $u := u_2$, $y := t_2$ and $t := t_3$, and the prime is the derivative with respect to $x(=t_1)$.

2.2. *The k-reduction*

The KP hierarchy is an unconstrained system in the sense that the dependent variables u_j are independent of each other in the Lax operator (2). This independence is not necessary: we can impose constraints between dependent variables without loss of consistency. The most familiar are the k-reductions of the KP hierarchy for an integer $k \geqslant 2$, for which the constraints are $\mathcal{L}_{\mathrm{KP}}^k = \mathcal{B}_k$, i.e. all the coefficients of negative powers in ∂ of $\mathcal{L}_{\mathrm{KP}}^k$ are zero,

$$\left(\mathcal{L}_{\mathrm{KP}}^k\right)_{-m} = 0 \tag{5}$$

where $m = 1, 2, \ldots$. This is equivalent to the t_{lk} independence of the system, for a natural number l. For example, the 2- and 3-reductions lead to the KdV and the Boussinesq hierarchy, respectively. For later purposes, we make the trivial remark that the reduction conditions (5) are compatible with the Lax equation: the conditions are invariant with respect to the time evolutions,

$$\begin{aligned}\left(\frac{\partial \mathcal{L}_{\mathrm{KP}}^k}{\partial t_n}\right)_{-m} &= \left(\left[\mathcal{B}_n, \mathcal{L}_{\mathrm{KP}}^k\right]\right)_{-m} \\ &= \left([\mathcal{B}_n, \mathcal{B}_k]\right)_{-m} \\ &= 0\end{aligned}$$

because the Hamiltonians have only derivatives, i.e. non-negative power terms in ∂.

2.3. *Supersymmetric extensions*

Supersymmetric extensions of the KP hierarchy (SKP) have been vigorously studied by both mathematicians and physicists. In particular, they appear in the context of superstring and/or quantum gravity theories [1]. The first supersymmetric extension was done by Manin and Radul [10], referred to as MRKP[3], in which the differential operator in superspace, i.e. the superderivative, and its inverse,

$$D := \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x} \qquad D^{-1} = \theta + \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial x}\right)^{-1} \tag{7}$$

[3] Another formulation of SKP is given in [11, 12].

play the parallel roles of ∂ and ∂^{-1} in the standard 'bosonic' KP. Here, θ is a Grassmann odd variable, accordingly the square of D turns out to be the ordinary derivative,

$$D^2 = \frac{\partial}{\partial x}. \tag{8}$$

In other words D can be regarded as a square root of ∂. According to the superspace formalism, superfields Φ_j play the role of dependent variables in the MRKP, whose Lax operator is defined as

$$\mathcal{L}_{\mathrm{MR}} = D + \sum_{j=1}^{\infty} \Phi_j D^{1-j}. \tag{9}$$

Besides the bosonic time variables $\mathbf{t}$, infinite fermionic time variables $(\tau_1, \tau_2, \ldots)$ must be introduced. Consequently, we observe that both the bosonic and fermionic time flows of the superfields make up a system of super-differential equations.

To make a comparison with another extension of the KP hierarchy considered in the following section, we exhibit the lowest degree bosonic time flows of the MRKP, which are given by the Lax equation of even order,

$$\frac{\partial \mathcal{L}_{\mathrm{MR}}}{\partial t_n} = [\mathcal{B}_{2n}, \mathcal{L}_{\mathrm{MR}}] \tag{10}$$

where the Hamiltonian is the standard one: $\mathcal{B}_{2n} := \left(\mathcal{L}_{\mathrm{MR}}^{2n}\right)_+$. In addition, there certainly exist fermionic time flows given by the odd order Lax equation, we do not need them, however, in the present consideration, for details see [10, 13]. One can show the lowest degree equation of (10) is an extension of the KP equation (4), which can be given in the component form [14]

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' + \frac{3}{2}v''v\right]' \tag{11a}$$

$$\frac{3}{4}\frac{\partial^2 v}{\partial y^2} = \left[\frac{\partial v}{\partial t} - \frac{1}{4}v''' - \frac{3}{2}(uv)'\right]' \tag{11b}$$

where the bosonic variable u and the fermionic one v are defined by $D\Phi_2 = v + \theta u$, and $t := t_3$ and $y := t_2$.

Besides the MRKP, various types of supersymmetric extension of the KP hierarchy are considered [15, 16]. For example, a non-standard Lax equation by Brunelli and Das [17] leads to an extension of the KP equation of the following form,

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' - \frac{3}{2}v''v - \frac{3}{2}v'\int^x \frac{\partial v}{\partial y}\,\mathrm{d}x - \frac{3}{2}v\frac{\partial v}{\partial y}\right]' \tag{12a}$$

$$\frac{3}{4}\frac{\partial^2 v}{\partial y^2} = \left[\frac{\partial v}{\partial t} - \frac{1}{4}v''' - \frac{3}{2}(uv)' - \frac{3}{2}u\int^x \frac{\partial v}{\partial y}\,\mathrm{d}x + \frac{3}{2}v'\int^x \frac{\partial u}{\partial y}\,\mathrm{d}x\right]' \tag{12b}$$

where, similar to the MRKP, u and v are bosonic and fermionic variables, respectively. In contrast to (11a) and (11b), there appear non-local terms in these coupled equations.

3. Extensions of the KP hierarchy by fractional-order integral operators

This section provides the extensions of the standard KP hierarchy by fractional-order integral operators, which is the main topic of the present paper.

3.1. Extended Lax operator

Recall that the Leibniz rule (1) is applicable when the order j of an 'integral' is an arbitrary real (or complex) number. It will be interesting to consider the case when the Lax operator includes fractional-order integrals, and then, to enquire whether the system gives a consistent hierarchy or not[4]. In the following consideration, we accept the axiom that the fractional-order integral operators exist and also the exponential law $\partial^{-i}\partial^{-j} = \partial^{-(i+j)}$ holds for fractional i and j, for a while. We will see the Riemann–Liouville integral of fractional order enjoys these requirements in the next section.

3.1.1. Extension by the half-order integrals. For the simplest case of an extension of the KP hierarchy, we consider the Lax operator including the half-order integrals in addition to the Lax operator (2). We restrict ourselves to the case that the highest order term is ∂ as in the KP. Accordingly, we define the most general half-order integral operator,

$$\mathcal{M}_{1/2} = v_3\partial^{-1/2} + v_5\partial^{-3/2} + v_7\partial^{-5/2} + \cdots \tag{13}$$

where v_m are the dependent variables of degree $m/2$. We have set the 'differentiation' term $\partial^{1/2}$ to be absent: this resulted from the Lax equation defined below. We remark that the Lax operator composed only of the half-order integrals (13) itself does not produce any consistent hierarchy, because its products do not close in the half-order integral operators: we need integer-order integral/differential operators to close the algebra. With this definition, we consider the following Lax operator,

$$\mathcal{L}_{1/2} = \mathcal{L}_{\mathrm{KP}} + \mathcal{M}_{1/2} \tag{14}$$

and the standard Lax equation for the flows with respect to the time $\mathbf{t} = (t_1, t_2, \ldots)$,

$$\frac{\partial \mathcal{L}_{1/2}}{\partial t_n} = [\mathcal{B}_n, \mathcal{L}_{1/2}]. \tag{15}$$

If we take the standard definition of the Hamiltonian, $\mathcal{B}_n := \left(\mathcal{L}_{1/2}^n\right)_+$, whose lower degree sequence is

$$\mathcal{B}_1 = \partial \tag{16a}$$

$$\mathcal{B}_2 = \partial^2 + 2v_3\partial^{1/2} + 2u_2 \tag{16b}$$

$$\mathcal{B}_3 = \partial^3 + 3v_3\partial^{3/2} + 3u_2\partial + 3(v_5 + v_3')\partial^{1/2} + 3u_3 + 3u_2' + 3v_3^2 \tag{16c}$$

then we find closed coupled nonlinear PDEs, an extended KP hierarchy by the half-order integrals, hereafter eKP$_{1/2}$. Other 'non-standard' definitions of $\mathcal{B}_n$ such as $\left(\mathcal{L}_{1/2}^n\right)_{\geqslant 1/2}$ cause inconsistency. To show the consistency of the system, we derive the lowest degree coupled PDE from (15), the extended KP equation by the half-order integral, i.e. the eKP$_{1/2}$ equation. Just as for the original KP equation (4), we need the first two non-trivial equations of (15). The coefficients of the negative powers in ∂ of

$$\frac{\partial \mathcal{L}_{1/2}}{\partial t_2} = [\mathcal{B}_2, \mathcal{L}_{1/2}] \tag{17}$$

are

$$\partial^{-1/2}\colon \frac{\partial v_3}{\partial y} = 2v_5' + v_3'' \tag{18a}$$

[4] Here we restrict ourselves to rational j; if j is irrational, the Lax equation could not give a closed system, see the following argument.

$$\partial^{-1}\colon \frac{\partial u_2}{\partial y} = 2u_3' + u_2'' + 2v_3v_3' \tag{18b}$$

$$\partial^{-3/2}\colon \frac{\partial v_5}{\partial y} = 2v_7' + v_5'' + 2(v_3u_2)' \tag{18c}$$

$$\partial^{-2}\colon \frac{\partial u_3}{\partial y} = 2u_4' + u_3'' + 2u_2u_2' + 3v_5v_3' + v_3v_5' - v_3v_3'' \tag{18d}$$

whereas those of

$$\frac{\partial \mathcal{L}_{1/2}}{\partial t_3} = [\mathcal{B}_3, \mathcal{L}_{1/2}] \tag{19}$$

are

$$\partial^{-1/2}\colon \frac{\partial v_3}{\partial t} = 3v_7' + 3v_5'' + v_3''' + 6(v_3u_2)' \tag{20a}$$

$$\partial^{-1}\colon \frac{\partial u_2}{\partial t} = 3u_4' + 3u_3'' + u_2''' + 6u_2u_2' + 6(v_3v_5)' + \frac{3}{2}\left(v_3'^2 + v_3v_3''\right). \tag{20b}$$

Eliminating the dependent variables u_3, u_4, v_5 and v_7, we find the coupled PDE with non-local term

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial u}{\partial t} - \frac{1}{4}u''' - 3uu' + \frac{3}{8}(v^2)'' - \frac{3}{4}v'\int^x \frac{\partial v}{\partial y}\,\mathrm{d}x - \frac{3}{4}v\frac{\partial v}{\partial y}\right]' \tag{21a}$$

$$\frac{3}{4}\frac{\partial^2 v}{\partial y^2} = \left[\frac{\partial v}{\partial t} - \frac{1}{4}v''' - 3(uv)'\right]' \tag{21b}$$

where $u := u_2$ and $v := v_3$. This coupled equation has never been known, to the present author's knowledge, so the system will be new. As expected, (21a) reduces to the KP equation (4) when v is absent. We observe the resemblance between (21a), (21b) and the MRKP equations (11a), (11b) or the non-standard SKP equations (12a), (12b), however, they are not exactly identical. This resemblance obviously comes from the fact that the derivative in superspace can be read as a square root of the derivative, which is formally equivalent to the feature of the half-order derivative $\partial^{1/2}$. In contrast to the supersymmetric extensions, the extension considered in this section works without using Grassmann numbers.

3.1.2. Extension by the $1/N$th-order integrals. Having observed the extension by the half-order integrals is successful, we now consider more generic extensions by the $1/N$th-order integrals ($N = 3, 4, \ldots$), eKP$_{1/N}$ hierarchies. In these cases, we need to introduce integral operators $\partial^{-1/N}, \partial^{-2/N}, \ldots, \partial^{-(N-1)/N}$ simultaneously to give a consistent Lax equation, since we have to close the commutator algebra in the Lax equations under the axiom $\partial^{-i}\partial^{-j} = \partial^{-i-j}$. For $N = p$, a prime number, there appears a new system coupled to the KP hierarchy. For example, we give an outline of the $N = 3$ case, in which the Lax operator should be made up of

$$\mathcal{L}_{1/3} = \mathcal{L}_{\mathrm{KP}} + \mathcal{M}_{1/3} + \mathcal{M}_{2/3} \tag{22}$$

where

$$\mathcal{M}_{1/3} = w_4\partial^{-1/3} + w_7\partial^{-4/3} + w_{10}\partial^{-7/3} + \cdots \tag{23}$$

$$\mathcal{M}_{2/3} = w_5\partial^{-2/3} + w_8\partial^{-5/3} + w_{11}\partial^{-8/3} + \cdots \tag{24}$$

and $\deg[w_m] = m/3$. We observe that the standard Lax equation and the definition of the Hamiltonian similar to the former case give a consistent hierarchy of coupled PDEs. One can see the lowest coupled PDE arises from the first two non-trivial Lax equations. The coefficients of ∂ in

$$\frac{\partial \mathcal{L}_{1/3}}{\partial t_2} = [\mathcal{B}_2, \mathcal{L}_{1/3}] \tag{25}$$

are

$$\partial^{-1/3}\colon \frac{\partial w_4}{\partial y} = 2w_7' + w_4'' \tag{26a}$$

$$\partial^{-2/3}\colon \frac{\partial w_5}{\partial y} = 2w_8' + w_5'' + \left(w_4^2\right)' \tag{26b}$$

$$\partial^{-1}\colon \frac{\partial u_2}{\partial y} = 2u_3' + u_2'' + 2(w_4 w_5)' \tag{26c}$$

$$\partial^{-4/3}\colon \frac{\partial w_7}{\partial y} = 2w_{10}' + w_7'' + \left(w_5^2\right)' + 2(w_4 u_2)' \tag{26d}$$

$$\partial^{-5/3}\colon \frac{\partial w_8}{\partial y} = 2w_{11}' + w_8'' + \frac{8}{3} w_7 w_4' + \frac{4}{3} w_4 w_7' + 2w_5 u_2' - \frac{2}{3} w_4 w_4'' + 2u_2 w_5' \tag{26e}$$

$$\partial^{-2}\colon \frac{\partial u_3}{\partial y} = 2u_4' + u_3'' + 2u_2 u_2' + \frac{10}{3} w_8 w_4' + \frac{4}{3} w_4 w_8' + \frac{8}{3} w_7 w_5' + \frac{2}{3} w_5 w_7' - \frac{4}{3} w_5 w_4'' - \frac{2}{3} w_4 w_5'' \tag{26f}$$

whereas those of

$$\frac{\partial \mathcal{L}_{1/3}}{\partial t_3} = [\mathcal{B}_3, \mathcal{L}_{1/3}] \tag{27}$$

are

$$\partial^{-1/3}\colon \frac{\partial w_4}{\partial t} = 3w_{10}' + 3w_7'' + w_4''' + 6(w_4 u_2)' + 3\left(w_5^2\right)' \tag{28a}$$

$$\partial^{-2/3}\colon \frac{\partial w_5}{\partial t} = 3w_{11}' + 3w_8'' + w_5''' + 6(w_5 u_2)' + 6(w_7 w_4)' + 2w_4'^2 + 2w_4 w_4'' \tag{28b}$$

$$\begin{aligned} \partial^{-1}\colon \frac{\partial u_2}{\partial t} &= 3u_4' + 3u_3'' + u_2''' + 6u_2 u_2' + 6(w_8 w_4)' + 6(w_7 w_5)' + w_5 w_4'' \\ &\quad + 3w_5' w_4' + 3w_4^2 w_4' + 2w_4 w_5''. \end{aligned} \tag{28c}$$

These are nine equations for the nine dependent variables so that we can combine them into the coupled PDE of u_2, w_4 and w_5.

For N being a composite number, we observe that the new system is coupled to the system coming from the prime factors of N. For example, the $\mathrm{eKP}_{1/4}$ system is a new system coupling to the $\mathrm{eKP}_{1/2}$ system given above.

Finally, we should remark that the introduction of a pseudo-derivative of irrational order does not make a finite closed system: we need an uncountable number of additional $\mathcal{M}$ such as (23) and (24).

3.2. The conservation laws

Since the $\mathrm{eKP}_{1/N}$ hierarchy is constructed within the framework of Lax formalism, we expect that the system is integrable *a priori*. In fact, we observe that there are infinite conservation laws, which can be derived by standard procedure [23] for the Lax operator under consideration $\mathcal{L}_{1/N}$, say, then we find

$$\frac{\partial}{\partial t_m}\mathrm{Res}\big(\mathcal{L}_{1/N}^n\big) = \frac{\partial}{\partial x}P_{m,n} \tag{29}$$

where the residue is defined as $\big(\mathcal{L}_{1/N}^n\big)_{-1}$ and $P_{m,n}$ is a differential polynomial of u_j and v_j. In appendix A, we give a proof to (29), where we see that the presence of the fractional-order integral operators does not modify formula (29). Hence, we expect the existence of many special solutions to the $\mathrm{eKP}_{1/N}$ hierarchies, just like the solitons in the original KP.

In addition to these conserved charges with integer degree, we have another set which has non-integer degree. For concreteness, we consider the $\mathrm{eKP}_{1/2}$, in which there exist conserved charges with degree $k + 1/2$ ($k = 0, 1, 2, \ldots$): we can find the charges come from

$$\mathrm{Res}\left(\mathcal{L}_{1/2}^{k+\frac{1}{2}}\right) \tag{30}$$

where the square root of the Lax operator is constructed by the usual procedure,

$$\mathcal{L}_{1/2}^{\frac{1}{2}} = \partial^{\frac{1}{2}} + \tfrac{1}{2}v_3\partial^{-1} + \tfrac{1}{2}u_2\partial^{-\frac{3}{2}} + \tfrac{1}{2}\left(v_5 - \tfrac{1}{4}v_3'\right)\partial^{-2} + \cdots. \tag{31}$$

Although we can construct the charges with half-integer degree, there does not exist a consistent time flow generated by the Hamiltonian $\mathcal{B}_{k+\frac{1}{2}} := \mathcal{L}_{1/2}^{k+\frac{1}{2}}$.

In general, for the $\mathrm{eKP}_{1/N}$ ($N \geqslant 3$) we will find the existence of additional sequences of conserved charges.

3.3. The k-reduction of the $eKP_{1/N}$

In this subsection, we make a comment on the k-reduction of the $\mathrm{eKP}_{1/N}$, the truncation of the t_{lk} flow. Unlike the standard KP hierarchy, the reduction condition $\mathcal{L}_{1/N}^k = \mathcal{B}_k$ does not work in the $\mathrm{eKP}_{1/N}$ hierarchies due to the property of fractional integrals. For, the compatibility (6) between the reduction condition and the Lax equation does not hold when the fractional order 'derivative' operators are present in the Hamiltonian $\mathcal{B}_k$. This comes from the fact that the Leibniz rule for non-integer order is not a finite sum even if the order j is positive; hence the right-hand side of the corresponding equation to (6) induces negative terms in ∂, i.e.

$$([\mathcal{B}_k, \mathcal{B}_{k'}])_{-\frac{m}{N}} \neq 0 \qquad (m = 1, 2, 3, \ldots). \tag{32}$$

For example, the commutator of the Hamiltonians $\mathcal{B}_2$ and $\mathcal{B}_3$ for the $\mathrm{eKP}_{1/2}$ gives

$$([\mathcal{B}_3, \mathcal{B}_2])_{-\frac{1}{2}} = \tfrac{3}{2}(v_3u_2')' - \left(v_3^3\right)' \neq 0 \tag{33}$$

as well as the coefficient of $\partial^{-m/2}$ ($m = 2, 3, \ldots$). Although it is not clear at present whether one can impose consistent reduction conditions on the Lax operator of the $\mathrm{eKP}_{1/N}$, we may consider the truncated system at hand.

Here, we present the remarkable fact that there exists an algebraic solution to the 2-reduction of the $\mathrm{eKP}_{1/2}$ equations (21*a*) and (21*b*), which should be referred to as the $\mathrm{eKdV}_{1/2}$ equation,

$$\frac{\partial u}{\partial t} = \frac{1}{4}u''' + 3uu' - \frac{3}{8}(v^2)'' \tag{34a}$$

$$\frac{\partial v}{\partial t} = \frac{1}{4}v''' + 3(uv)'. \tag{34b}$$

Note the strong resemblance between the supersymmetric KdV equation [21] and (34*a*), (34*b*). The solution is an extension of the rational solution to the KdV equation:

$$u(x,t) = -\frac{5}{16}\frac{1}{(x+ct)^2} + \frac{c}{3} \tag{35a}$$

$$v(x,t) = \pm\frac{\sqrt{165}}{24}\frac{1}{(x+ct)^{3/2}} \tag{35b}$$

where c is a constant with $\deg[c] = 2$, which is required by keeping the correct degree of the dependent variables. It is an important future subject to find a systematic procedure for constructing regular solutions other than the singular one obtained here.

4. The Riemann–Liouville integrals of fractional order

So far, we constructed the extensions of the KP hierarchy by introducing the pseudo-differential operator of fractional order, the fractional integral. However, we have treated such operators as only generators of a non-commutative algebra. In the construction of the standard KP hierarchy, we use the Leibniz rule of negative-order derivatives (1), which can be realized by the iterative use of integration by parts, e.g., for $j = 1$,

$$\begin{aligned}\partial^{-1}(fg) = \int^x fg\,\mathrm{d}x &= fG^{(1)} - \int^x f'G^{(1)}\,\mathrm{d}x \\ &= fG^{(1)} - f'G^{(2)} + \int^x f''G^{(2)}\,\mathrm{d}x \\ &= fG^{(1)} - f'G^{(2)} + f''G^{(3)} - \cdots \\ &= \sum_{k=0}^{\infty}(-1)^k f^{(k)}G^{(k+1)}\end{aligned} \tag{36}$$

where $G^{(k)}$ is the k times indefinite integral of g. The cases of higher order j can be shown by multiplicative operation of (36). Hence, we accept the statement that the negative-order derivative operator is equivalent to the indefinite integral operator.

For the case of fractional j, how can we realize formula (1)? To find the appropriate fractional integration/derivation on a function assumed to be existing in the last section, we recall the Riemann–Liouville integral of order $\alpha \in \mathrm{C}$ for $\mathrm{Re}\,\alpha > 0$,

$$I^\alpha f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-z)^{\alpha-1} f(z)\,\mathrm{d}z \tag{37}$$

where $f(x)$ is assumed to be locally integrable and rapidly decreasing on the lower boundary a. We realize that when $\mathrm{Re}\,\alpha < 0$, $\alpha \notin -\mathbb{N}$, (37) should be read as

$$I^\alpha f(x) := I^{\alpha+n} f^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} I^{\alpha+n} f(x) \tag{38}$$

where n is the first integer of $n + \mathrm{Re}\,\alpha > 0$. The assumption for $f(x)$ guarantees the last equality in (38), which means that the Riemann–Liouville integral commutes with the ordinary derivative. Hence, we understand that definition (37) can be extended by analytic continuation to the whole α, for details see appendix B. One can also show, when α tends to an integer, the Riemann–Liouville integral turns out to be the ordinary integration/differentiation, as it should be. In addition, we will observe in appendix B that the Riemann–Liouville integral complies with the exponential or additive index law, $I^\alpha I^\beta = I^{\alpha+\beta}$, which is assumed in the derivation of the $\mathrm{eKP}_{1/N}$ hierarchy.

We exhibit the action of the Riemann–Liouville integral on some functions,

$$I^{\alpha}\, \mathrm{e}^{\lambda x} = \frac{1}{\lambda^{\alpha}}\, \mathrm{e}^{\lambda x} \tag{39}$$

where $\lambda > 0$ and,

$$I^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} x^{\mu+\alpha} \tag{40}$$

where the operand, x^{μ}, is defined as zero if $x < 0$.

In appendix B, we observe that definitions (37) and (38) lead to the Leibniz rule (1) for fractional order,

$$I^{\alpha}(fg) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} f^{(k)}(I^{k+\alpha} g) \tag{41}$$

where

$$\binom{-\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!}. \tag{42}$$

Note that (41) turns out to be (36) as α tends to 1, obviously. In this respect, we should remark the fact that the Riemann–Liouville integrals do not define the Leibniz rule (41) uniquely: further generalization to the rule is also possible [6]. However, it is sufficient for the present purpose that the Riemann–Liouville integrals can fulfil (41). With the features observed above, we accept that the Riemann–Liouville integrals yield the pseudo-differential operators of fractional order,

$$\partial^{-\alpha} = I^{\alpha}. \tag{43}$$

Thus, we find that the pseudo-differential operator of fractional order in the Lax operator of the $\mathrm{eKP}_{1/N}$ hierarchy is not only an element of a non-commutative algebra, but also a concrete operator on a certain class of functions.

Although we do not need the explicit operation of the fractional integrals in the derivation of the extended KP hierarchies considered in the previous section, we expect that the direct application of the Riemann–Liouville integral (37) is required in further consideration of the $\mathrm{eKP}_{1/N}$ system. For, we will need the Riemann–Liouville integrals when we consider the analytic solutions to the $\mathrm{eKP}_{1/N}$ hierarchies through, e.g., the inverse scattering method, Bäcklund transformation, Painlevé analysis and so on [22].

5. Concluding remarks

In this paper, we have considered the extensions of the KP hierarchy by introducing fractional-order integral/differential operators, the $\mathrm{eKP}_{1/N}$. In particular, if we introduce the half-order integral operator, we find the resulted $\mathrm{eKP}_{1/2}$ hierarchy is analogous to the SKP hierarchies. Other extensions, the $\mathrm{eKP}_{1/N}$ hierarchies, are also considered and the fact that there exist infinite conserved currents is observed. We have also found an algebraic solution to the $\mathrm{eKdV}_{1/2}$ equation, the reduced $\mathrm{eKP}_{1/2}$ equation.

To obtain a deep understanding of the $\mathrm{eKP}_{1/N}$ hierarchies, it is profitable to make an analysis in the Sato formulation, and also the tau-function formulation [4], just as in the case of the standard KP hierarchy and the SKP hierarchies [18–20]. Apart from the whole hierarchies, it will be interesting to investigate simply the integrability of the coupled equations such as (21*a*) and (21*b*) through the Painlevé analysis [24]. Another approach is also attainable: the Hirota bilinear method is applicable to find special solutions such as solitons. As mentioned in the last section, it will be necessary to consider the Riemann–Liouville integrals explicitly for these analyses.

Incidentally, apart from integrable systems there are many works on fractional-order evolution equations for relaxation, diffusion, stochastic process and so on [9]. Although we have obtained the nonlinear PDEs with normal derivative through the application of fractional calculus, it is intriguing to formulate a systematic procedure for creating the PDEs with anomalous derivative, i.e. fractional-order derivative. A possibility will be given by the application of another 'Leibniz rule' in the formulation given in this paper; in fact, the Riemann–Liouville integrals enjoy miscellaneous types of 'Leibniz rule' as mentioned in the last section.

Appendix A

In this appendix we give the proof to (29) for the eKP$_{1/N}$ hierarchy. First of all we have

$$\frac{\partial}{\partial t_m}\mathrm{Res}\big(\mathcal{L}^n_{1/N}\big) = \mathrm{Res}\left(\frac{\partial}{\partial t_m}\mathcal{L}^n_{1/N}\right) = \mathrm{Res}\big(\big[\mathcal{B}_m, \mathcal{L}^n_{1/N}\big]\big). \tag{A.1}$$

On the right-hand side, each term in the bracket is of the form $f\partial^p g\partial^q - g\partial^q f\partial^p$, where f and g are differential polynomials of dependent variables and p and q are, in general, the multiples of $1/N$ admitting negative values. Now we observe by the Leibniz rule,

$$f\partial^p g\partial^q = f\sum_{j=0}^{\infty}\binom{p}{j} g^{(j)}\partial^{q+p-j} \tag{A.2}$$

so the contribution of this term to the residue occurs only when $q+p$ is an integer with $q+p \geqslant -1$. For p and q enjoying these conditions, we find

$$\mathrm{Res}(f\partial^p g\partial^q) = \binom{p}{p+q+1} fg^{(p+q+1)} \tag{A.3}$$

where $p+q+1$ is a non-negative integer so that the right-hand side is well defined. Similar consideration for $g\partial^q f\partial^p$ leads to

$$\begin{aligned}\mathrm{Res}([f\partial^p, g\partial^q]) &= \binom{p}{p+q+1} fg^{(p+q+1)} - \binom{q}{p+q+1} f^{(p+q+1)}g \\ &= \binom{p}{p+q+1} fg^{(p+q+1)} - (-1)^{p+q+1}\binom{p}{p+q+1} f^{(p+q+1)}g \\ &= \binom{p}{p+q+1}\{fg^{(p+q+1)} - (-1)^{p+q+1} f^{(p+q+1)}g\}\end{aligned} \tag{A.4}$$

where the fact

$$\binom{\alpha}{j} = (-1)^j\binom{j-1-\alpha}{j} \tag{A.5}$$

for $\alpha \in \mathrm{C}$ is used. Thus we find only the cases $p+q+1 \in \mathbb{N}$ contribute; however, we easily observe

$$fg^{(k)} - (-1)^k f^{(k)}g = \frac{\partial}{\partial x}\left\{\sum_{j=0}^{k-1}(-1)^j f^{(j)}g^{(k-j-1)}\right\} \tag{A.6}$$

for any $k \in \mathbb{N}$. We, therefore, conclude that all the contribution to the residue on the right-hand side of (A.1) is total derivative. In this proof, obviously, the number n in (A.1) is not restricted to integer: we can make sequences of conserved charges such as (30) other than integer sequences.

Appendix B

This appendix presents some of the important properties of the Riemann–Liouville integrals (37).

First of all, we observe the exponential law $I^\alpha I^\beta = I^{\alpha+\beta}$. For $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$,

$$\begin{aligned} I^\alpha I^\beta f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \mathrm{d}z (x-z)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_a^z \mathrm{d}w (z-w)^{\beta-1} f(w) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \mathrm{d}w\, f(w) \int_z^x \mathrm{d}z (x-z)^{\alpha-1} (z-w)^{\beta-1} \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x \mathrm{d}w (x-w)^{\alpha+\beta-1} f(w) \\ &= I^{\alpha+\beta} f(x) \end{aligned} \tag{B.1}$$

where we changed the integration order and used the fact that the z-integral in the second line was expressed by the beta function. This formula can be extended to the case when one or both of $\mathrm{Re}\,\alpha$ and $\mathrm{Re}\,\beta$ are negative, by (38) and the commutativity of I^α and ordinary derivative.

Next we derive the Leibniz rule for fractional-order integral/differential operator (41). For $\mathrm{Re}\,\alpha > 0$, if we expand one of the operands, say, f in Taylor series, we find

$$\begin{aligned} I^\alpha (f(x)g(x)) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} f(z) g(z)\, \mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)}(x) (x-z)^k g(z)\, \mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)}(x) \int_a^x (x-z)^{k+\alpha-1} g(z)\, \mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} f^{(k)}(x) \Gamma(k+\alpha) I^{k+\alpha} g(x) \\ &= \sum_{k=0}^{\infty} \binom{-\alpha}{k} f^{(k)}(x) I^{k+\alpha} g(x) \end{aligned} \tag{B.2}$$

which can be extended to $\mathrm{Re}\,\alpha < 0$ by ordinary differentiation (38).

Finally, we comment on the Riemann–Liouville integral of order $\mathrm{Re}\,\alpha < 0$, i.e. a differentiation of arbitrary order. In definition (38) the integral is well defined; however, if we simply put $\mathrm{Re}\,\alpha < 0$ in the definition of I^α (37), then the definition turns out to be a divergent integral whatever the function f is, due to the singularity of the kernel $(x-z)^{\alpha-1}$ at the upper bound. To give a well-defined meaning for the divergent integral, we can take the finite part of it, the Pf (*partie finie*) prescription. To see this we consider only the case $-1 < \mathrm{Re}\,\alpha < 0$ for simplicity, the case of $\mathrm{Re}\,\alpha < -1$ can be treated similarly. We now define the 'regularized integral' with a positive cut-off parameter ϵ as

$$\begin{aligned} I^\alpha_\epsilon f(x) &:= \frac{1}{\Gamma(\alpha)} \int_a^{x-\epsilon} (x-z)^{\alpha-1} f(z)\, \mathrm{d}z \\ &= \frac{1}{\Gamma(\alpha)} \int_\epsilon^{x-a} s^{\alpha-1} f(x-s)\, \mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha+1)} \left\{ s^\alpha f(x-s) \Big|_\epsilon^{x-a} + \int_\epsilon^{x-a} s^\alpha f'(x-s)\, \mathrm{d}s \right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha+1)}\left\{-\epsilon^{\alpha} f(x-\epsilon) + \int_{\epsilon}^{x-a} s^{\alpha} f'(x-s)\,\mathrm{d}s\right\} \\
&= \frac{1}{\Gamma(\alpha+1)}\left\{-\epsilon^{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} f^{(k)}(x)\epsilon^{k} + \int_{\epsilon}^{x-a} s^{\alpha} f'(x-s)\,\mathrm{d}s\right\}
\end{aligned} \tag{B.3}
$$

due to the condition $f(a) = 0$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. In the last line of (B.3), when we take $\epsilon \to 0$, the terms of $\epsilon^{\alpha+k}$ in the infinite sum are zero if $k \geqslant 1$, and also the integral term remains finite. Thus we can perform the Pf prescription by the following definition:

$$
\begin{aligned}
I^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)}\mathrm{Pf}\int_{a}^{x}(x-z)^{\alpha-1} f(z)\,\mathrm{d}z \\
&:= \lim_{\epsilon\to 0}\left\{I_{\epsilon}^{\alpha} f(x) - \frac{(-1)}{\Gamma(\alpha+1)}\epsilon^{\alpha} f(x)\right\} \\
&= \frac{1}{\Gamma(\alpha+1)}\int_{0}^{x-a} s^{\alpha} f'(x-s)\,\mathrm{d}s \\
&= I^{\alpha+1} f'(x).
\end{aligned} \tag{B.4}
$$

We, therefore, conclude that definition (38) is well defined and the Riemann–Liouville integral can be continued analytically to negative $\mathrm{Re}\,\alpha$.

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